Properties of 3×3 Magic Squares

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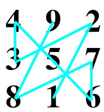
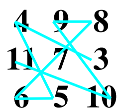
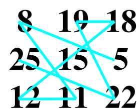
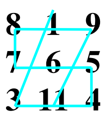
Consider the following magic squares[1] with lines connecting the values in ascending order.







A curious pattern appears, which raises the question of whether the same polygonal pattern appears in all magic squares.[2] The following magic square tells us that that the answer to this question is no.



In fact, there are only 2 possible patterns. In order to verify this claim, we define requisite notation, introduce several preliminary properties, and present the main result in Property 3. Let the 9 unique values of a 3×3 magic square be denoted as $a_1 < a_2 < \ldots < a_9$. The position values are denoted as x_{ij}, i.e.,

$$\begin{matrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{matrix}$$

and $S = x_{11} + x_{12} + x_{13}$ denotes the *magic constant*.

[1] Nine unique positive numbers arranged in a square matrix such that all row sums, column sums, and diagonal sums are equal.
[2] This question was raised in the August 1997 issue of *Scientific American* (see p. 88).

Property 1. $a_5 = x_{22} = S/3$.

Proof.[3]

$$x_{11} + x_{21} + x_{31} = S \Rightarrow x_{21} + x_{31} = S - x_{11}$$
$$x_{11} + x_{22} + x_{33} = S \Rightarrow x_{22} + x_{33} = S - x_{11}$$
$$x_{13} + x_{23} + x_{33} = S \Rightarrow x_{23} + x_{33} = S - x_{13}$$
$$x_{13} + x_{22} + x_{31} = S \Rightarrow x_{22} + x_{31} = S - x_{13}$$

Therefore, $x_{21} + x_{31} = x_{22} + x_{33}$ and $x_{23} + x_{33} = x_{22} + x_{31}$. Adding these equations:

$$x_{21} + x_{31} + x_{23} + x_{33} = 2x_{22} + x_{33} + x_{31} \text{ or } x_{21} + x_{23} = 2x_{22}.$$

Adding x_{22} to each side of the equation gives,

$$x_{21} + x_{22} + x_{23} = 3x_{22}.$$

But $x_{21} + x_{22} + x_{23} = S$, which implies $x_{22} = S/3$.

In order to show that $a_5 = x_{22}$, note that $x_{11} + x_{22} + x_{33} = S$ and $x_{22} = S/3$ implies that either $x_{11} < x_{22} < x_{33}$ or $x_{11} > x_{22} > x_{33}$. Similarly, $x_{13} < x_{22} < x_{31}$ or $x_{13} > x_{22} > x_{31}$, $x_{21} < x_{22} < x_{23}$ or $x_{21} > x_{22} > x_{23}$, and $x_{12} < x_{22} < x_{32}$ or $x_{12} > x_{22} > x_{32}$. Therefore, there are 4 values smaller than x_{22} and 4 values larger than x_{22}, which implies $a_5 = x_{22}$. □

Property 2. $a_1 + a_9 = a_2 + a_8 = a_3 + a_7 = a_4 + a_6 = 2a_5$.

Proof. Due to Property 1, for any $i \in \{1,2,3,4,6,\ldots,9\}$ there exists some $j \in \{1,2,3,4,6,\ldots,9\}\backslash\{i\}$ such that $a_i + a_5 + a_j = S$. Consider $i = 1$ and suppose $j \neq 9$. Then $S = a_1 + a_5 + a_j \leq a_1 + a_5 + a_8 < a_1 + a_5 + a_9 < a_9 + a_5 + a_k$ for $k \neq 1$. But when $i = 9$, there exits some $k \in \{2,3,4,6,\ldots,8\}$ such that $a_9 + a_5 + a_k = S$, which contradicts the possibility $j \neq 9$. Therefore, $a_1 + a_5 + a_9 = S = 3a_5$, or $a_1 + a_9 = 2a_5$. The same arguments can be applied for other values of i (i.e., consider $i = 2$ and suppose $j \notin \{8,9\}$, consider $i = 3$ and suppose $j \notin \{7,8,9\}$, etc.). □

Corollary 1. The values of a_1, a_2, a_3, a_4, a_6, a_7, a_8, and a_9 are symmetric with respect to a_5, i.e., $a_5 - a_1 = a_9 - a_5$, $a_5 - a_2 = a_8 - a_5$, $a_5 - a_3 = a_7 - a_5$, and $a_5 - a_4 = a_6 - a_5$.

Proof. Follows directly from Property 2 (e.g., $a_1 + a_9 = 2a_5$ can be rewritten as $a_5 - a_1 = a_9 - a_5$). □

[3] Property 1 is a slight extension of the property $x_{22} = S/3$; the proof of $x_{22} = S/3$ appears in *The Moscow Puzzles* by B.A. Kordemsky (Dover Publications, 1992. p. 292).

Property 3. A 3×3 magic square is 1 of 2 possible forms.

Form 1:				Form 2:		
a_7	a_1	a_8		a_6	a_1	a_8
a_6	a_5	a_4		a_7	a_5	a_3
a_2	a_9	a_3		a_2	a_9	a_4

Specific magic squares can be composed according to the following rules.

Form 1:

1. Arbitrarily select values for Δ_1 and Δ_2.
2. Arbitrarily select a value for a_5 that satisfies $a_5 > 3\Delta_1 + 2\Delta_2$.
3. Set the remaining values according to:

$a_6 = a_5 + \Delta_1$, $\quad a_4 = a_5 - \Delta_1$
$a_7 = a_6 + \Delta_2$, $\quad a_3 = a_4 - \Delta_2$
$a_8 = a_7 + \Delta_1$, $\quad a_2 = a_3 - \Delta_1$
$a_9 = a_8 + \Delta_1+\Delta_2$, $\quad a_1 = a_2 - (\Delta_1+\Delta_2)$.

Accordingly, form 1 can be rewritten as:

$a_5 + (\Delta_1 + \Delta_2)$	$a_5 - (3\Delta_1 + 2\Delta_2)$	$a_5 + (2\Delta_1 + \Delta_2)$
$a_5 + \Delta_1$	a_5	$a_5 - \Delta_1$
$a_5 - (2\Delta_1 + \Delta_2)$	$a_5 + (3\Delta_1 + 2\Delta_2)$	$a_5 - (\Delta_1 + \Delta_2)$

Form 2:

1. Arbitrarily select values for Δ_1 and Δ_2.
2. Arbitrarily select a value for a_5 that satisfies $a_5 > 3\Delta_1 + \Delta_2$.
3. Set the remaining values according to:

$a_6 = a_5 + \Delta_1$, $\quad a_4 = a_5 - \Delta_1$
$a_7 = a_6 + \Delta_2$, $\quad a_3 = a_4 - \Delta_2$
$a_8 = a_7 + \Delta_1$, $\quad a_2 = a_3 - \Delta_1$
$a_9 = a_8 + \Delta_1$, $\quad a_1 = a_2 - \Delta_1$.

Accordingly, form 2 can be rewritten as:

$a_5 + \Delta_1$	$a_5 - (3\Delta_1 + \Delta_2)$	$a_5 + (2\Delta_1 + \Delta_2)$
$a_5 + (\Delta_1 + \Delta_2)$	a_5	$a_5 - (\Delta_1 + \Delta_2)$
$a_5 - (2\Delta_1 + \Delta_2)$	$a_5 + (3\Delta_1 + \Delta_2)$	$a_5 - \Delta_1$

Proof. We may limit our consideration to the possibilities of $a_1 = x_{11}$ and $a_1 = x_{12}$ (cases 1 and 2 below). This is because the form of a magic square does not substantively change when it is transposed (i.e., columns become rows and rows become columns), columns 1 and 3 are interchanged, or rows 1 and 3 are interchanged.

Case 1: $a_1 = x_{11}$

From Property 2, it follows that $a_9 = x_{33}$. This means that $x_{31} + x_{32} = x_{13} + x_{23} = a_1 + a_5$. But this implies that x_{31}, x_{32}, x_{13}, x_{23} all must be between a_1 and a_5. This is impossible because there are only 3 such values. Hence $a_1 \neq x_{11}$, and a form corresponding to case 1 does not exist.

Case 2: $a_1 = x_{12}$

From Property 2, it follows that $a_9 = x_{32}$. This means that $x_{31} + x_{33} = a_1 + a_5$, which implies that x_{31} and x_{33} are between a_1 and a_5. As a form is invariant when columns are interchanged, we may assume without loss of generality that $x_{31} < x_{33}$. This leads to 3 possibilities that we consider in turn.

Case 2a: $a_1 = x_{12}, x_{31} = a_3, x_{33} = a_4$

From Property 2, $x_{31} = a_3$ implies $x_{13} = a_7$, and $x_{33} = a_4$ implies $x_{11} = a_6$. Up to this point, the magic square appears as:

a_6 a_1 a_7
a_5
a_3 a_9 a_4

From $a_6 + x_{21} + a_3 = a_7 + x_{23} + a_4$, it follows that $x_{21} = a_8$ and $x_{23} = a_2$. But $a_6 + a_8 + a_3 < a_7 + a_2 + a_4$. Hence, a form corresponding to case 2a does not exist.

Case 2b: $a_1 = x_{12}, x_{31} = a_2, x_{33} = a_3$

From Property 2, $x_{31} = a_2$ implies $x_{13} = a_8$, and $x_{33} = a_3$ implies $x_{11} = a_7$. From $a_7 + x_{21} + a_2 = a_8 + x_{23} + a_3$, it follows that $x_{21} = a_6$ and $x_{23} = a_4$, and the magic square appears as:

a_7 a_1 a_8
a_6 a_5 a_4
a_2 a_9 a_3

Note that $a_1 + a_7 + a_8 = a_2 + a_6 + a_7$ implies

$$a_8 - a_6 = a_2 - a_1$$

and $a_1 + a_7 + a_8 = a_2 + a_3 + a_9$ implies

$$(a_9 - a_8) + (a_2 - a_1) = (a_7 - a_3).$$

But from Corollary 1, it follows that $a_2 - a_1 = a_9 - a_8$ and $(a_7 - a_3) = 2(a_7 - a_5)$. Therefore,

$$a_8 - a_6 = a_9 - a_8 = a_7 - a_5.$$

Furthermore, $a_8 - a_6 = a_7 - a_5$ implies $a_8 - a_7 = a_6 - a_5$. Letting $\Delta_1 = a_6 - a_5$ and $\Delta_2 = a_7 - a_6$ we find $a_8 - a_7 = \Delta_1$ and $a_9 - a_8 = \Delta_1 + \Delta_2$, which leads to the rules for composing a magic square that matches form 1.

Case 2c: $a_1 = x_{12}, x_{31} = a_2, x_{33} = a_4$

From Property 2, $x_{31} = a_2$ implies $x_{13} = a_8$, and $x_{33} = a_4$ implies $x_{11} = a_6$. From $a_6 + x_{21} + a_2 = a_8 + x_{23} + a_4$, it follows that $x_{21} = a_7$ and $x_{23} = a_3$, and the magic square appears as:

a_6 a_1 a_8
a_7 a_5 a_3
a_2 a_9 a_4

Note that $a_2 + a_4 + a_9 = a_3 + a_4 + a_8$ implies

$$a_9 - a_8 = a_3 - a_2$$

and $a_1 + a_6 + a_8 = a_3 + a_4 + a_8$ implies

$$a_6 - a_4 = a_3 - a_1.$$

But from Corollary 1, it follows that $a_3 - a_2 = a_8 - a_7$, $a_6 - a_4 = 2(a_6 - a_5)$, and $a_3 - a_1 = a_9 - a_7$. Therefore,

$$a_9 - a_8 = a_8 - a_7 \text{ and}$$

$$a_9 - a_7 = 2(a_6 - a_5).$$

Furthermore, $a_9 - a_8 = a_8 - a_7$ implies $a_9 - a_7 = 2(a_9 - a_8) = 2(a_6 - a_5)$, or $a_9 - a_8 = a_6 - a_5$. Letting $\Delta_1 = a_6 - a_5$ and $\Delta_2 = a_7 - a_6$ we find $a_8 - a_7 = \Delta_1$ and $a_9 - a_8 = \Delta_1$, which leads to the rules for composing a magic square that matches form 2. ☐

Two Examples

Form 1: $\Delta_1 = 1$, $\Delta_2 = 2$. $a_5 = 10$

13	**3**	**14**
11	**10**	**9**
6	**17**	**7**

Form 2: $\Delta_1 = 1$, $\Delta_2 = 2$. $a_5 = 6$

7	**1**	**10**
9	**6**	**3**
2	**11**	**5**