What Does Operating a Newsstand Have to do with the Risk-Neutral Density of a Future Stock Price?

A Little Background

A few years ago I was thinking about a property of the newsvendor model[1] and its relevance for inferring the probability distribution of a future stock price from current option prices. I put these thoughts down on paper and, after reviewing the literature, discovered that this issue is far from new. The linkage between option prices and future stock price probabilities has been well known since Ross (1976). Breeden and Litzenberger (1978) observe that the second derivative of a function defining the value of a European call option expiring in t periods with strike price s is the risk-neutral probability density function of the stock price in t periods, i.e., traders using the density function would value options at current market prices.[2] Since this time, researchers have proposed and tested different methods for estimating the density function from the market value of options for a finite set of strike prices (e.g., Aït-Sahalia and Lo 1998, Banz and Miller 1978, Jackwerth and Rubinstein 1996, Jarrow and Rudd, Longstaff 1995, Madan and Milne 1994, Rubinstein 1994, Shimko 1993). For example, Rubinstein (1994) and Jackwerth and Rubinstein (1996) formulate as a problem of minimizing the distance from prior probabilities subject to constraints of correct market prices whereas Shimko (1993) and Aït-Sahalia and Lo (1998) propose methods for estimating the second derivative of an option value function.

The following note uses a different mechanism to infer the probability distribution, but as I suspect it is essentially equivalent to the second derivative mechanism, the approach adds virtually nothing to what is already known. Perhaps of most interest is that the mechanism highlights a close relationship between two "different" models in the literature—newsvendor ordering and option valuation.

Aït-Sahalia, Y. and A.W. Lo, "Nonparametric Estimation of State-Price Densities Implicit in Financial Asset Prices," *Journal of Finance*, 53 (1998), 499-547.

Banz, R. and M. Miller, "Prices for State-Contingent Claims: Some Estimates and Applicationss," *Journal of Business*, 51 (1978), 653-672.

Breeden, D. and R. Litzenberger, "Prices of State-Contingent Claims Implicit in Options Prices," *Journal of Business*, 51 (1978), 621-651.

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[1] The newsvendor model is an abstraction of the problem of deciding how many newspapers to purchase. Demand is stochastic with a known probability distribution function. Money is lost if too many papers are ordered (e.g., purchase cost less salvage value) and money is lost if too few papers are ordered (e.g., lost profit). The objective is to maximize expected profit. Given excess cost rate c_e, shortage cost rate c_s, and probability distribution $F(x)$, the optimal order quantity is $F^{-1}(c_s/(c_s+c_e))$.

[2] The expected value of a European call option expiring in t periods with strike price s, risk-free interest rate r, and with $f(x)$ denoting the probability density function of stock price in t periods is

$C(\mathrm{s}) = e^{-rt}\int_s^{\infty}(x-s)f(x)dx.$ Taking the first and second derivative with respect to strike price yields

$C'(s) = e^{-rt}[F(s) + 1]$ and $C''(s) = \ \ e^{-rt}f(s)$. Therefore, $f(s) = e^{rt}C''(s)$.

Jackwerth, J.C. and M. Rubinstein, "Recovering Probability Distrubitons from Contemporary Security Prices," *Journal of Finance*, 51 (1996), 1611-1631.
Jarrow, R. and A. Rudd, "Approximate Option Valuation for Arbitrary Stochastic Processes," *Journal of financial Economics*, 10 (1982), 347-369.
Longstaff, F. "Option Pricing and the Martingale Restriction," *Review of Financial Studies*, 8 (1995), 1091-1124.
Madan, D. B. and F. Milne, "Contingent Claims Valued and Hedged by Pricing and Investing in a Basis," *Mathimatical Finance*, 4 (1994), 223-245.
Ross, S., "Options and Efficiency," *Quarterly Journal of Economics*, 90 (1976), 75-89.
Rubinstein, M., "Implied Binomial Trees," *Journal of Finance*, 49 (1994), 771-818.
Shimko, D., "Bounds of Probabilities," *RISK*, 6 (1993), 33-37.

The Market Distribution of a Future Stock Price

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July 1999

Abstract

What is the probability distribution of the price of a publicly traded asset at some future point in time? Individual traders will likely have different answers to this question, but in total, the composite of their beliefs is a single probability distribution. Knowledge of this distribution can prove useful as implausible cases suggest opportunity for profit. This note shows how a property governing the minimum sum of left and right linear loss integrals can be used to deduce this composite probability distribution. A key feature of the property is that it holds for all continuous probability distributions, meaning that no assumptions on the underlying stochastic process of asset price are required.

Property and Analysis

Let X denote a random variable that has continuous distribution $F(x)$ on $\Omega \subset \mathbf{R}$ with density $f(x)$, and let

$$g(x,\alpha) = \alpha E[\max\{0, X - x\}] + (1 - \alpha)E[\max\{0, x - X\}]. \tag{1}$$

The following property holds for any continuous probability distribution.

Property 1. For any $\alpha \in [0, 1]$,

$$x^*(\alpha) = \text{argmin}_{x\in\Omega}\{g(x,\alpha)\} \tag{2}$$

if and only if

$$F(x^*(\alpha)) = \alpha. \tag{3}$$

Proof. (1) can be written as $g(x,\alpha) = \alpha\int_{\{t\in\Omega|t>x\}}(t - x)f(t)dt + (1 - \alpha)\int_{\{t\in\Omega|t\leq x\}}(x - t)f(t)dt$. Thus, the first order condition is

$$\partial g/\partial x = -\alpha[1 - F(x)] + (1 - \alpha)F(x) = 0, \tag{4}$$

which, by virtue of isomorphic mapping $F:\Omega\rightarrow[0,1]$, admits a unique solution $x^*(\alpha)$ satisfying $F(x^*(\alpha)) = \alpha$. Further, $\partial^2 g/\partial x^2 = f(x) \geq 0$ (g is convex in x), $\partial g(\inf\Omega,\alpha)/\partial x = -\alpha \leq 0$, and $\partial g(\sup\Omega,\alpha)/\partial x = 1 - \alpha \geq 0$. If $\partial g(\inf\Omega,\alpha)/\partial x = 0$, then $x^*(\alpha) = \inf\Omega$ by (2) and $\partial g(\sup\Omega,\alpha)/\partial x > 0$; similarly if $\partial g(\sup\Omega,\alpha)/\partial x = 0$, then $x^*(\alpha) = \sup\Omega$ by (2) and $\partial g(\inf\Omega,\alpha)/\partial x < 0$. Therefore, the first order condition yields the global minimum at $x^*(\alpha)$. ❑

Suppose that X is the price of one share of stock t periods into the future. The current prices of options for different strike prices expiring in t periods reflect the potentially disparate beliefs of many traders on the underlying distribution of X. The *market distribution* of X is the composite of all these beliefs, i.e., traders using the market distribution would value options at current prices. Property 1 can be used to deduce the market distribution from observed prices of European call and put options under one condition: *the market value of an option is proportional to the expected pay-out upon expiration.*

First, consider the idealized setting where options are traded at every strike price $x \in \Omega$. Let $C(x)$ and $P(x)$ be the current prices of European call and put options at strike price x and expiration date t periods from now. Notice that (3) implies that the mapping from α to $x^*(\alpha)$ defined in (2) is an isomorphism (i.e., not only is there a unique $x \in \Omega$ satisfying (2) for each $\alpha \in [0,1]$ but there is also a unique $\alpha \in [0,1]$ satisfying (2) for each $x \in \Omega$). Letting $\alpha^*(x)$ denote the inverse of $x^*(\alpha)$, it follows from (3) that

$$F(x) = \alpha^*(x). \quad (5)$$

Thus, for any observed x, $C(x)$, and $P(x)$, the value of $F(x)$ is the unique value of α satisfying

$$g(x,\alpha) = \alpha C(x) + (1 - \alpha)P(x) = \text{argmin}_{y\in\Omega}\{\alpha C(y) + (1 - \alpha)P(y)\}. \quad (6)$$

For example, if from observed data at a given instant we see that $g(x,.25)$, $g(x,.50)$, and $g(x,.75)$ are minimized at strike prices \$85, \$90, and \$95, then the first, second, and third quartiles of the market distribution are \$85, \$90, and \$95.

The characterization of the market distribution is more coarse when $C(x)$ and $P(x)$ are observable for only a handful of different strike prices, but the same principles apply. Let $0 < x_1 < x_2 < \ldots < x_n$ denote strike prices for market traded European call and put options expiring in t periods and define $x_0 = 0$, $x_{n+1} = \infty$,

$$\Psi_j = \{\alpha \mid g(x_j,\alpha) < \min_{k\neq j}\{\ g(x_k,\alpha)\}\}, \quad (7)$$

$$\Phi_\alpha = (x_{j-}, x_{j+}), \quad (8)$$

where

$$j- = \min\{j \mid j = \text{argmin}_k\{g(x_k,\alpha)\}\} - 1, \quad (9)$$

$$j+ = \max\{j \mid j = \text{argmin}_k\{g(x_k,\alpha)\}\} + 1 \quad (10)$$

(note that the set—$\{j \mid j = \text{argmin}_k\{g(x_k,\alpha)\}\}$—is contiguous due to the convexity of g). From Property 1, it follows that

$$F(x_j) \in \Psi_j \quad (11)$$

$$F^{-1}(\alpha) \in \Phi_\alpha. \quad (12)$$

For example, if from observed data at a given instant we see that $g(90,\alpha)$ is smaller than $g(85,\alpha)$ and $g(95,\alpha)$ when $\alpha \in [.35, .65]$, then $F(90) \in [.35, .65]$. Alternatively, $F^{-1}(\alpha) \in (85, 95)$ for any $\alpha \in [.35, .65]$.

Summary and Implications

I have described a nonparametric methodology for characterizing the market distribution of the future price of a stock as of a given time instant. The methodology relies on a simple and fundamental structural property of a convex combination of left and right linear loss integrals. Implausible characterizations are indicative of inconsistent market valuation and therefore signal the potential for arbitrage profits.